Finding periodic points from short time series

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(Received 30 October 1996)

We present an algorithm for finding low-order periodic points of chaotic maps from possibly very short time series. No information about the map other than the time series is used. The method finds all the periodic points of a piecewise linear approximation of the map. We present examples showing the effectiveness of the method for the Henon and Ikeda maps and a chaotic electronic circuit, including a ''cycle expansion'' calculation of the Hausdorff dimension for the Henon map. [S1063-651X(97)11307-1]

PACS number(s): 05.45.+b

I. INTRODUCTION

We are interested in estimating low-order periodic points of (possibly chaotic) maps where the only information we have is a short time series generated by iteration of the map. Many techniques for analyzing chaotic maps require knowledge of the low-order periodic points [1,2], but often little is said about how to obtain them. Knowledge of the periodic points is fundamental to ''periodic orbit theory'' [2], which is a much more elegant approach to calculations of dynamical invariants than the traditional Monte Carlo methods. Previous methods relied on knowing the map and/or having very long time series available [1,3–6]; our method is intended for use when the map is unknown and only a short time series is available. We will find approximate periodic points of the true map by finding all the exact periodic points of a given order for a piecewise linear approximation to the map. The partition for the piecewise linear approximation is defined by triangulating the data [7].

II. TRIANGULATION AND APPROXIMATION

First we make some definitions. For $n+1$ points $v_0, \ldots, v_n$, in general position in R^n, their convex hull is a (closed) *n-simplex* σ. That is,

$$\sigma = \left\{ \sum_i \lambda_i v_i : \ \lambda_i \in R, \ \lambda_i \geqslant 0 \ \ \forall i, \quad \text{and} \ \sum_i \lambda_i = 1 \right\}.$$

The points $v_0, \ldots, v_n$ are called the *vertices* of σ and we denote the simplex by $\sigma = \sigma(v_0, \ldots, v_n)$. Let C be a closed convex set in R^n. G is a *triangulation* of C if G is a collection of n simplices whose union is C and the intersection of two simplices is either the empty set or an m simplex, $m<n$. Note that the boundary of an n-simplex σ (for $n>0$) consists of $(n-1)$ simplices, which we call *faces* of σ.

There are many triangulations for a given set of points. One of the more useful is the Delaunay triangulation [8], which has the property that it minimizes the average diameter of the simplices. This minimizes the average error in our piecewise linear approximation, as discussed below. The Delaunay triangulation is discussed in detail in [9] and efficient algorithms for computing it are described in [10]. Although easy to calculate, the Delaunay triangulation is not necessarily the best approximation to the map. Methods such as the ''data dependent triangulations'' [11] might be better. However, in our numerical experiments to date, there has been no useful difference between the Delaunay and other triangulations.

We say that $\lambda_i = \lambda_i(x)$, $i = 0,1, \ldots, n$, are the *barycentric coordinates* for x with respect to v_i, if

$$x = \sum_i \lambda_i v_i \quad \text{and} \quad 1 = \sum_i \lambda_i,$$

where $x \in \sigma(v_o, \ldots, v_n)$. A piecewise linear approximation to $f: R^n \to R^n$ is then given by

$$\hat{f}(x) = \sum_i \lambda_i f(v_i).$$

We note [7] that $\hat{f}$ is continuous in x; if f is C^2 then $\hat{f}(x) = f(x) + O(\Delta^2)$, where Δ is the diameter of σ and if f is affine, then $\hat{f} = f$.

Suppose that $\hat{f}^j(x)$ lies in the simplex $\sigma^{k_j}(v_0^{k_j}, \ldots, v_n^{k_j})$, with barycentric coordinates $\lambda^{k_j} = (\lambda_0^{k_j}, \ldots, \lambda_n^{k_j})^T$. We define the piecewise linear approximation to the $j+1$st iterate of the map by

$$\hat{f}^{j+1}(x) = \sum_i \lambda_i^{k_j} f(v_i^{k_j}). \tag{1}$$

Note that this approximation depends on only the *first* image of the vertices under f; we *could* have used the jth images of the vertices but this is a significantly worse approximation to f^j than Eq. (1).

The point here is that we can model the map itself reasonably well, but not higher iterates of the map, especially for chaotic systems. In this sense, iterating the model is more sensible than trying to model the iterates of the map. Because of this, methods for finding periodic points that rely on modeling the higher iterates of chaotic maps are in a sense inherently unreliable. Numerical methods using models of only the first iterate of the map have been used to estimate the invariant measure of an attractor [12] and the largest Lyapunov exponent [13].

III. FINDING APPROXIMATE PERIODIC POINTS

We assume that we have a time series $X=\{x_i: x_i \in R^n;\ i=1,\ldots,t+1\}$. We can take our vertices to be $v_i=x_i$ and their images $f(v_i)=x_{i+1}$ for $i=1,\ldots,t$. If the set $\{x_i\}$ is a scalar time series embedded using a time-delay embedding [14,15], then $n-1$ components of the map f will be linear and our approximation is exact for these components.

If the x_i are corrupted by noise we need to apply some form of noise reduction before we look for periodic points. One way to do this is to model f from the data and use the model to give values of $f(v_i)$ for vertices v_i. Examples of this where we use triangulations to build the model and then find periodic points are described in [16,17].

To find a periodic point of period p, we need to solve

$$x=\hat{f}^p(x), \quad \text{that is,} \quad \sum_i \lambda_i^0 v^0=\sum_i \lambda_i^{p-1} f(v_i^{p-1}). \quad (2)$$

However, we also require that each intermediate step be consistent. This means that we require that the barycentric coordinates of the kth image in σ^k be those used to calculate the $(k+1)$th image. That is,

$$\sum_i \lambda_i^k v^k=\hat{f}^k(x)=\sum_i \lambda_i^{k-1} f(v_i^{k-1}), \quad (3)$$

in addition to $\Sigma_i \lambda_i^k=1,\ \forall k=0,\ldots,p$.

The problem now is finding an appropriate sequence of simplices, $\sigma^{k_i},\ i=0,\ldots,p;\ k_i\in\{1,\ldots,N\}$ (we have N simplices in our triangulation), in order to construct Eqs. (2) and (3). We do this by constructing a ''transition matrix'' T, where

$$T_{ij}=\begin{cases}1 & \text{if} \quad f(\sigma^i)\cap\sigma^j\neq\varnothing\\ 0 & \text{otherwise,}\end{cases}$$

where $i,j=1,\ldots,N$. Given a starting index k_0, we can recursively search T for sequences where

$$T_{k_0,k_1}T_{k_1,k_2}\cdots T_{k_{p-1},k_0}=1,$$

so that the index sequence $k_0,k_1,\ldots,k_{p-1}$ corresponds to a sequence of simplices that *might* contain a period p point.

We have a system of linear equations in $p(n+1)$ unknowns (the λ_i^k) for a period p point in R^n. We have $p(n+1)$ equations so, barring degeneracies, the solution is unique. In order for it to correspond to an approximate periodic point, we require that $\lambda_i^k\geqslant 0,\ \forall k,\ \forall i$. The system of equations is moderately sparse and the sparseness increases as p increases. In fact there are $p(n+1)(2n+1)$ nonzero entries out of $p^2(n+1)^2$ in total. In matrix form, we can write the equations as follows. Let $1_m=(1,1,\ldots,1)$, a row vector of length m. Let $V^i=(v_0^i,\ldots,v_n^i)$ and $f(V^i)=(f(v_0^i),\ldots,f(v_n^i))$; these are $n\times(n+1)$ matrices. Then we have

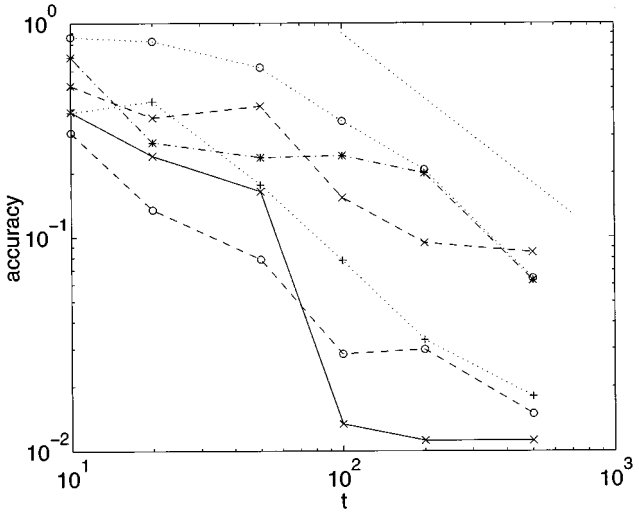

FIG. 1. Mean accuracy vs length of time series for periods 1 to 6 of the Ikeda map. The dotted line without markers indicates the scaling law given in the text.

$$\begin{pmatrix} 1_{n+1} & 0 & \cdots & 0\\ 0 & 1_{n+1} & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & 1_{n+1}\\ V^0 & 0 & \cdots & -f(V^{p-1})\\ -f(V^0) & V^1 & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & V^{p-1}\end{pmatrix}\begin{pmatrix}\lambda^0\\ \vdots\\ \lambda^{p-1}\end{pmatrix}=\begin{pmatrix}1_p^T\\ 0\end{pmatrix}, \quad (4)$$

where 0 denotes a zero matrix of appropriate size.

Our algorithm can be described as follows: (1) Find the set of vertices $\{v_i: i=1,\ldots,t\}$ and their images $\{f(v_i)\}$. (2) Triangulate the vertices giving the collection of simplices $\{\sigma^i: i=1,\ldots,N\}$ and calculate the transition matrix T. (3) Choose a period p. Set $m=1$. (4) For the starting simplex σ^m, do the following: (i) Recursively search T to find sequences of simplices that might contain periodic points. Reject sequences that contain simplices σ^i where $i<m$ as these sequences have already been checked. (ii) For each sequence, construct the system of linear equations [Eq. (4)] and solve them. If all the λ_i^k are non-negative, the solution defines an approximate periodic point. This point generates $p-1$ other period p points by iteration of the approximate map. (5) Set $m\leftarrow m+1$. If $m\leqslant N$, return to step 4.

The average volume of a simplex in a triangulation built from t vertices scales like $1/t$. In d dimensions, the volume also scales like ϵ^d where ϵ is the diameter of the simplex. Combining these, we have $\epsilon\sim t^{-1/d}$. The accuracy of the periodic point, defined by $\text{acc}_p(x)=\|f^p(x)-x\|$, scales as $\epsilon^2\sim t^{-2/d}$. So in R^2, the accuracy scales as $1/t$, in R^3 as $t^{-2/3}$, and so on. We also expect the accuracy of the periodic point to scale as e^p. Figure 1 shows the scaling of $\text{acc}_p(x)$ for different p and t for the Ikeda map. The points plotted are averages over each period p.

For the application of this technique to chaotic time series, we have ignored some subtleties. We have assumed

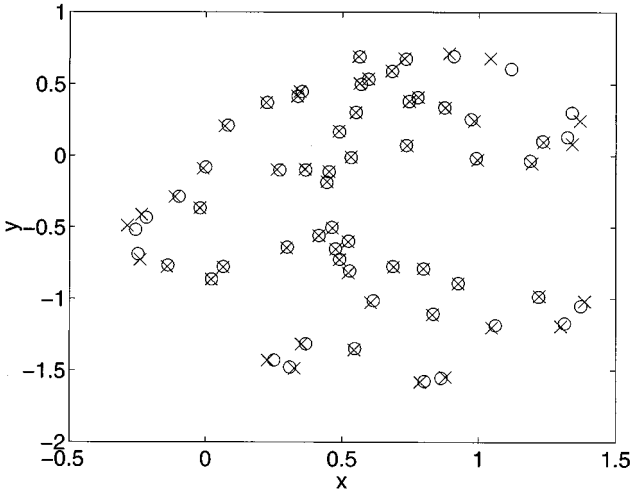

FIG. 2. Approximate periodic points for the Ikeda map (circles) and the true points (crosses).

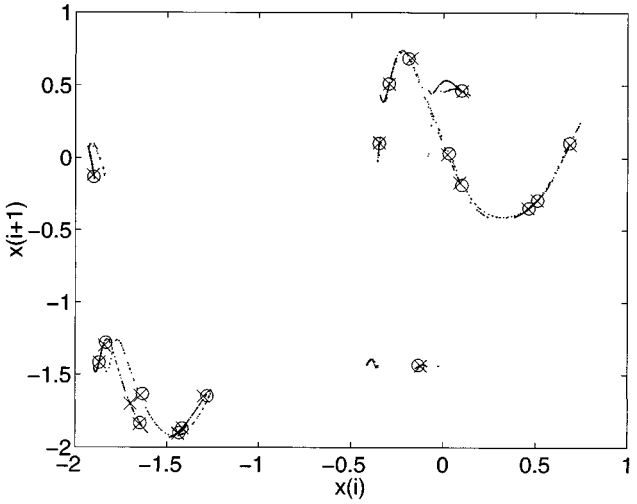

FIG. 3. Approximate periodic points (circles) and true points (crosses) up to period 3 for the electronic circuit, calculated using the 500 data points shown as dots. The scalar time series $x(i)$ is embedded with a simple lag-1 time delay embedding in R^2 to give points $[x(i),x(i+1)]$.

that, as t increases, the additional points are distributed uniformly across the convex hull of the attractor, leading to the simplex volume scaling as $1/t$. In fact, the additional points will be distributed according to the invariant density of the attractor and the actual scaling will depend on how often the trajectory returns to a neighborhood of the fixed point. In particular, points in frequently visited parts of the attractor will be approximated better.

Calculating the Delaunay triangulation and finding T both have complexity $O(t^2)$, as does finding the simplex sequences. For period p, the number of sequences is $O(te^p)$; however, we make some savings by rejecting previously checked sequences as described above. This pruning reduces the number of sequences by a factor of $1/p$. We solve a linear system of size $p(n+1)$, which is an operation of $O(p^3)$.

To illustrate the time needed, for 100 points of data from the Ikeda map, it took 2.925 s to calculate T and a total of 135 s to find all the periodic points up to period 6, of which 102 s were needed just for period 6. These calculations were run using MATLAB on a Silicon Graphics Indy with 100 MHz MIPS R4000 CPU running IRIX 5.3.

IV. EXAMPLES

Figure 2 shows periodic points from periods 1 to 6 calculated from a time series of 500 points of the Ikeda map,

$$z=x+iy\mapsto\delta+\beta z\exp\left(i\gamma-\frac{i\alpha}{1+|z|^2}\right),$$

with $\alpha=5.4$, $\beta=0.9$, $\gamma=0.4$, and $\delta=0.92$. Note the excellent agreement with the true periodic points shown. It is true that, in general, the number of periodic points of the true map is not necessarily equal to that for the piecewise linear approximation. However, for low-order points and sufficient data, equality will hold. Our numerical experiments indicate that any ''extra'' approximate periodic points tend to disappear with a change in the number of nearby data points. In this sense, spurious additional points are unstable and easily detected. Similarly, ''missing'' periodic points appear, and remain, once sufficient data points exist in the neighborhood to enable accurate modeling of the map.

We applied our technique to a time series obtained by numerical integration of equations that describe the chaotic dynamics of an electronic circuit [18]. Successive maxima of one component of the three-dimensional time series were used as a scalar time series $\{x_i: i=1,\ldots,t\}$, which (for this system) corresponds to points on a Poincaré section for the flow. This scalar time series is embedded in two dimensions and we look for periodic points of the map describing the evolution of these embedded points. Such periodic points correspond to periodic orbits of the original system. Figure 3 shows the 500 point data set used and some of the real and approximate periodic points up to period 3. We have not shown all the points up to period 3 as extracting the periodic orbits from the original system is a somewhat tedious task. For the points shown we note the good agreement with our approximate points, with the exception of a fixed point (period 1) at $(-1.7,-1.7)$, which our method failed to find. Our explanation of this is to observe that there is a gap in the data around $(-1.7,-1.7)$ and that the map is *extremely* complicated in this region. In other words, there are simply not enough nearby data points to accurately model that region.

To illustrate one application of our method we calculated the approximate periodic points, up to period 7, for the Hénon map

$$(x,y)\mapsto(1-1.4x^2+y,0.3x),$$

using only 50 data points. These periodic points have been used to calculate the Hausdorff dimension using the ''cycle expansion'' method [2]. As we will present details elsewhere of the combination of triangulation approximations and cycle expansions, we give only brief results here. The Hausdorff dimension is $D_H=1+D_s$ where D_s is the solution to

$$0=\prod_{P\in\mathcal{P}}[1-(\Lambda_P)^{D_s}],$$

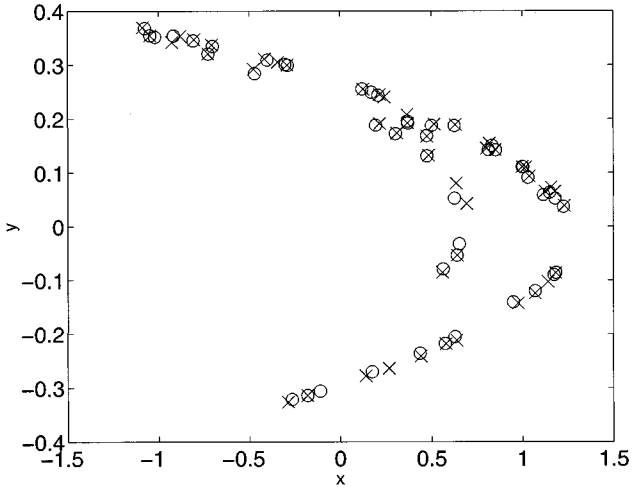

FIG. 4. Approximate periodic points up to period 7 for the Hénon map (circles) and the true points (crosses), calculated using 50 data points.

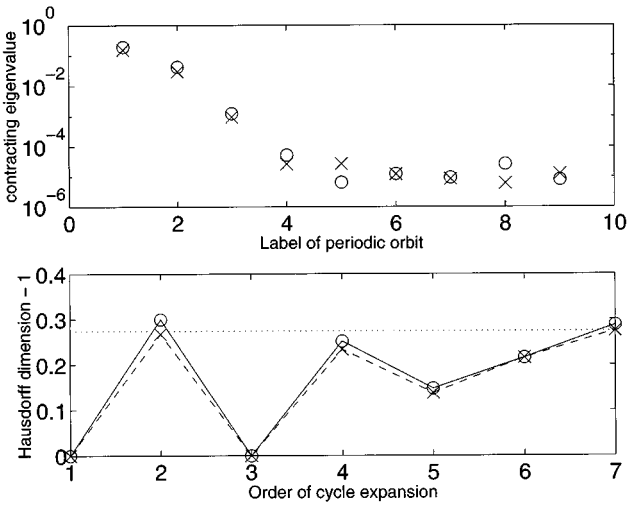

FIG. 5. Top axes show the values of Λ_P for the periodic orbits up to period 7 of the Hénon map. Circles are the approximate points, crosses are the true values. The bottom axes show $D_s = D_h - 1$ for cycle expansions up to order 7. The solid line is for the approximate periodic points, the dashed line is the true map. The dotted line shows the accepted value of $D_s \approx 0.274$.

and Λ_P is the absolute value of the contracting (stable) eigenvalue of the periodic orbit labeled by P, and $\mathcal{P}$ is the set of all ''prime'' cycles (see Ref. [2]). This infinite product is truncated to include particular combinations of low-order periodic orbits, the so-called ''cycle expansion.'' Figure 4 shows the true and approximate periodic points up to period 7. Figure 5 shows the values of Λ_P for both the true map and the approximate map for orbits up to period 7, and D_s as a function of the order of the cycle expansion. Our method reproduces the number, positions, and stability of the periodic points to good accuracy and this is reflected in the good agreement of the dimension calculation between the true and approximate maps.

We emphasize that the dimension estimates shown in Fig. 5 were obtained from only 50 data points with no other information about the map being used. The cycle expansion methods enable the calculation of other dynamical invariants such as the largest Lyapunov exponent and topological entropy. We have obtained good preliminary results with other maps and other invariant quantities.

V. DISCUSSION AND CONCLUSIONS

As our examples illustrate, our method is able to find the low-order period points from very short time series to good accuracy with reasonable computational effort. Our implementation is far from optimal and useful improvements could be made with a more efficient implementation.

Any method for *approximating* periodic points will involve a model of some kind. This is usually implicit in the method; we have chosen to make our model (piecewise linear over triangulations) explicit. This makes it possible to perform analysis of the model as if it were the true system. For example, we can easily calculate the Jacobian at the periodic points, for use with cycle expansions and this was illustrated with the example of Hausdorff dimension estimates for the Hénon map. It is intended to present a detailed analysis of this model class elsewhere.

ACKNOWLEDGMENTS

This research was partially supported by a grant from the Australian Research Council. The authors wish to thank the University of Western Australia for leave and financial support. We thank Henry Abarbanel and the Institute of Nonlinear Science at the University of California, San Diego, for providing us with the electronic circuit data.

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